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THE COMBINATORICS OF ALTERNATIVE FACTORIZATIONS

BY

ELSA FISMAN AND MARY SCHAPS

ABSTRACT

In this paper we count the number of distinct toroidal morphisms to 3-space satisfying Danilov's condition (0), obtaining the simple formula $M_n = 3 \cdot 4^{n-1}$ for n blowings up. We also give average and extremal values for the number of alternative factorization sequences for such a morphism.

Introduction

In algebraic geometry, one of the obstacles to generalizing results about surfaces to higher dimensional varieties is the existence of alternative factorizations of birational morphisms as a sequence of the elementary operations known as blowings up. Each blowing-up is an operation performed locally at a "center" of codimension at least two. For surfaces such centers are points and necessarily disjoint; for higher dimensional varieties the centers can intersect. In order to analyze the essential difficulty, "local" alternative factorizations at such intersection points, we must first eliminate the "noise" of "global" alternative factorizations, arising from blowing-up disjoint centers in varying orders. These "global" alternative factorizations exist already in the surface case, and we will study them in morphisms of 3-folds of a special type in which Danilov has shown the factorization to be locally unique. To simplify combinatorial analysis, we will work only with toroidal morphisms, in which the number of possible morphisms of a given type is finite. Since each such morphism can be completely represented by a graph in the affine plane, we will not actually use any algebraic geometry in proving the results.

The graphs we treat will be triangularizations of a basic triangle G_0 in R^3 with vertices P_1, P_2, and P_3, which form an affine basis for the plane containing the triangle. Each point $R = t_1P_1 + t_2P_2 + t_3P_3$ with $t_1 + t_2 + t_3 = 1$ will be assigned affine coordinates (t_1, t_2, t_3).

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DEFINITION. A triangularization of the basic triangle will be called a Farey graph if all the vertices have rational coordinates, and for each simplex R_1, R_2, R_3, the determinant of the three coordinate vectors is $\pm 1/d_1d_2d_3$, where d_i is the lowest common denominator of the coordinates of R.

The results and methods of this paper are combinatorial, involving the enumerations of certain types of Farey graphs. However, since the significance of this enumeration lies in algebraic geometry, we will pause briefly to describe the algebro-geometric object associated to each Farey graph.

To each simplex $\sigma = [R_1R_2R_3]$ we associate a copy X of affine three space with coordinates t_1, t_2, t_3. We regard $[R_1, R_2, R_3]$ as the dual graph of the configuration of coordinate planes in this space. Each vertex R_i corresponds to the plane $\{t_i = 0\}$, each edge $[R_i, R_j]$ corresponds to the axis $\{t_i = t_j = 0\}$ and the interior of the simplex corresponds to the origin $\{t_1 = t_2 = t_3 = 0\}$. Since the correspondence is a dual one, dimensions and inclusions are reversed.

If σ_0 is the basic triangle $P_1P_2P_3$, with coordinate functions x, y, z, then the affine coordinates of $R_1R_2R_3$ determine a mapping $f: X_\sigma \to X_{\sigma_0}$. If $d_iR_i = (a_i, b_i, c_i)$ is the minimal integral vector in the direction of R_2, then we set

$$x = t_1^{a_1}t_2^{a_2}t_3^{a_3}, \qquad y = t_1^{b_1}t_2^{b_2}t_3^{b_3}, \qquad z = t_1^{c_1}t_2^{c_2}t_3^{c_3}.$$

Since, by the definition of a Farey graph, the matrix of exponents has determinant ± 1, this matrix has an integral inverse and thus $f_\sigma^{-1}: X_{\sigma_0} \to X_\sigma$ is a rational map of the form

$$t_1 = x^{d_1}y^{e_1}z^{f_1}, \qquad t_2 = x^{d_2}y^{e_2}z^{f_2}, \qquad t_3 = x^{d_3}y^{e_3}z^{f_3},$$

where the exponents are integral but may be negative. f_σ^{-1} is well defined whenever x, y and z are all non-zero and maps this set U_{σ_0} isomorphically to the open set U_σ of X_σ on which all the t_i are non-zero. By composing this isomorphism we get birational correspondences among all the X_σ.

We now wish to glue the X together in a way compatible with the morphisms, so that each vertex of the original Farey graph will correspond to a unique divisor. We consider a second simplex $\sigma' = [R_1R_2R_4]$ which shares a common edge with σ. Since R_4 lies on the side of $[R_1, R_2]$ opposite to R_3, we have

$$d_3R_3 = \alpha_1(d_1R_1) + \alpha_2(d_2R_2) + \alpha_4(d_4R_4),$$

with α_4 negative. If A is the matrix of exponents defining f_σ and A' is the matrix of exponents defining $f_{\sigma'}$, a simple substitution of the formulae for f_σ in those for $f_{\sigma'}^{-1}$ shows that the correspondence $f_{\sigma'}^{-1} \circ f$ is given by the integral matrix of exponents $A'^{-1} \cdot A$. Since the columns of A are simple combinations of the columns of A', we obtain the formulae

$$t_1' = t_1 t_3^{\alpha_1}, \qquad t_2' = t_2 t_3^{\alpha_2}, \qquad t_4' = t_3^{\alpha_4}.$$

The α_i must therefore be integers, and by the symmetry between σ and σ' we conclude that $\alpha_3 = -1$, i.e. that

$$t_4' = t_3^{-1}.$$

Thus $X_\sigma - \{t_3 = 0\} \xrightarrow{\sim} X_{\sigma'} - \{t_4' = 0\}$. When the two spaces are glued together on this open set, the t_3 and t_4' axis map to the same P^1. The sets $\{t_i = 0\}$ and $\{t_i' = 0\}$ for $i = 1, 2$ map to the same irreducible divisors in the glued space intersecting along that P'. Thus the union of the simplices is still the dual graph of the system of special divisors.

When all the X_σ are glued together in this manner, we obtain a space X_F, together with a regular map $f_F : X_F \to X_{\sigma_0}$. Now let us give a description in terms of the Farey graph for the algebro-geometric operation of blowing-up one of the triple points or double curves in the special system of divisors.

I. Blowing up a simplex $\sigma = [R_1 R_2 R_3]$

To "blow-up" a simplex we replace it by the following subdivision: We add one new vertex

$$R = \frac{1}{d_1 + d_2 + d_3}(d_1 R_1 + d_2 R_2 + d_3 R_3),$$

and three edges $[RR_i]$ for $i = 1, 2, 3$. They create three new simplices of the form $\sigma_{ij} = [R_i R_j R]$.

We now consider the effect on the affine space X with coordinate functions $t_1 t_2 t_3$. If $(a_i, b_i, c_i) = v_i$ are the integral coordintes $d_i R_i$ of R_i, then $v_1 + v_2 + v_3 = v$ are the integral coordinates $(d_1 + d_2 + d_3)R$, and have no common divisor since

$$\det \begin{bmatrix} v_1 \\ v_2 \\ v \end{bmatrix} = \det \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \pm 1.$$

Consider the subsimplex $\sigma_{12} = [R_1 R_2 R]$. If we write the matrix of coefficients $A = [v_1^t v_2^t v_3^t]$, then

$$A_{12} = v_1^t v_2^t v^t = A \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$A_{12}^{-1}=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}A^{-1}.$$

Since A_{12}^{-1} is the matrix of coefficients for $t_1't_2't_3'$ with respect to x, y, z and similarly A^{-1} is the matrix of coefficients for $t_1t_2t_3$ with respect to x, y, z, we find that

$$t_1'=t_1t_3^{-1},\qquad t_2'=t_2t_3^{-1},\qquad t_3'=t_3.$$

The new divisor corresponding to the vertex R is then locally defined by $t_3=0$, and has a neighborhood with coordinates $t_1t_3^{-1}$, $t_2t_3^{-1}$, and ring of functions $k[t_1/t_3, t_2/t_3]$. When the three simplices σ_{12}, σ_{23}, and σ_{31} are glued together in the manner described above, we get a new space $X_{\{\sigma_{12},\sigma_{23},\sigma_{31}\}}$ which is isomorphic to X_σ except over the point $t_1=t_2=t_3=0$. At that point we have a new divisor obtained by glueing together three affine planes with coordinate functions t_i/t_k, t_j/t_k, giving the classical construction of the projective plane P^2. This is the algebro-geometric operation of blowing-up a point.

II. Blowing-up an edge $[R_1R_2]$

To "blow-up" an edge $[R_1R_2]$ we add the vertex

$$R=\frac{1}{d_1+d_2}(d_1R_1+d_2R_2).$$

If $\sigma=[R_1R_2R_3]$ is a simplex containing R_1R_2 as an edge, then add the edge $[RR_3]$ and replace σ by two simplices $\sigma_{13}=[R_1RR_3]$ and $\sigma_{23}=[R_2RR_3]$.

In the corresponding algebraic affine space X_σ with coordinates $t_1t_2t_3$, this corresponds to removing the t_3 axis C and replacing it by a fiber bundle $C\times P^1$. The new space can be covered by two affine neighborhoods, one with coordinates t_1, t_2/t_1, t_3, and one with coordinates t_1/t_2, t_2, t_3, patched together in such a way that the t_1/t_2 axis becomes a P^1.

Danilov has just settled affirmatively what we will call the weak factorization conjecture for Farey graphs: Any Farey graph can be obtained from any other by a sequence of blowings-up and their inverses, blowings-down. However, what we may call the strong factorization conjecture remains open: Given two Farey graphs F and F' can we find a third graph F'' which can be obtained from each by blowings-up alone? In other words, do the Farey graphs form a directed system under the partial ordering $\leqq$ determined by the operation of blowing up?

The general problem still appears quite intractable. We are restricting our attention temporarily to the special case in which F and F' are themselves obtained by blowings-up from the basic simplex, i.e. $\sigma_0 \leqq F, F'$. Hironaka has asked: given a minimal F'' such that $F, F' \leqq F''$, can we find a bound on the length of a chain connecting F to F'' as a function of the lengths of the chains connecting F and F' to σ_0?

In this way we are led to a study of the various alternative factorizations of $F'' \geqq \sigma_0$ by sequences of blowings-up. Considering the lattice of graphs F for which $F \leqq F''$ and $F \geqq \sigma_0$ we find that it is highly branched. Unfortunately, the greater part of that branching is of a trivial nature.

DEFINITION. Two factorizations of F'' by blowings-up will be called *locally equivalent* if the sequence of subdivisions of any simplex along the chain is the same for both factorizations, and only the global order in the choice of centers for the blowings-up is the same.

We are actually interested only in comparing lengths for sequences which are locally non-equivalent, and would thus like to replace each factorization sequence by its equivalence class under local equivalence. We would thus like to replace factorization sequences by some sort of branched factorization tree which would represent the entire equivalence class.

Although we do not yet have an entirely satisfactory solution in the general case, we have managed to give a complete solution in the special case of Farey graphs satisfying the following condition:

DEFINITION. A Farey graph will be called "exterior" if all the added points are on the edges of the basic triangle.

As a consequence of the main theorem of Danilov [1], any exterior Farey graph is factorizable by blowings-up, and any two factorization sequences are locally equivalent. In this paper we show how to associate to any exterior Farey graph a binary tree, how to count the number of exterior Farey graphs associated to each tree, how to deduce the total number of exterior graphs containing n added points, and how to count the number of locally equivalent alternative factorizations associated to a given tree.

We begin by counting the total number of factorization sequences.

CLAIM. The number of possible factorization sequences producing an exterior Farey graph in n steps is $(n+2)!/2$.

For $n = 1$, we blow up one of the three exterior edges of the basic triangle. If there are $k+2$ exterior edges after step $n = k-1$, then step $n = k$ will produce $k+3$ exterior edges. The total number of choices is thus

$$3 \cdot 4 \cdot \cdots \cdot (n+2) = (n+2)!/2.$$

We will show that most graphs have many factorizations, but there are wide variations in the number of alternative factorizations of a graph. The possibilities range from "deep" graphs with a unique factorization to "shallow" graphs, which have a much larger than average number of factorization sequences. In Fig. 1 we give the simplest case of an alternative factorization.

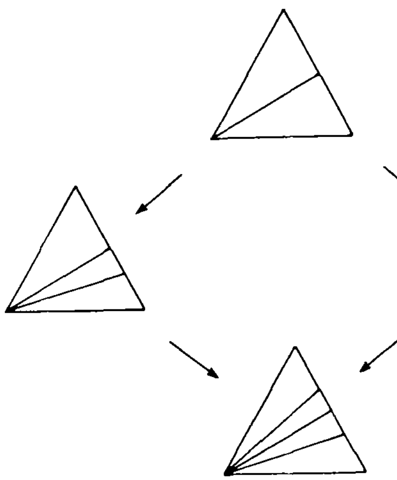


Fig. 1.

By the Danilov result quoted above, each equivalence class of factorization sequences under local equivalence corresponds to a unique exterior Farey graph F. Our main result is the following:

PROPOSITION. *The number M_c of distinct exterior Farey graphs obtained by c blowings up, for $c \geqq 1$, is $3 \cdot 4^{c-1}$. The set of such graphs can be partitioned into classes labelled by binary graphs, each class containing graphs with the same number and types of alternative factorizations.*

PROOF. An affine mapping of a triangular Farey subgraph of $R_1R_2R_3$ of a Farey graph F onto a triangular Farey subgraph $R'_1R'_2R'_3$ of a Farey graph F' will be called a Farey map if it is induced by the linear transformation carrying d_iR_i to $d'_iR'_i$. It is one of the convenient facts about polygonal complexes that such a mapping transforms a sequence of blowings up in $R_1R_2R_3$ to the corresponding sequence of blowings up in $R'_1R'_2R'_3$. To check this is a trivial exercise in affine geometry.

DEFINITION. Let S_c be the number of exterior Farey graphs with all added points on a single side, the side opposite P_1. Let N_c be the number of exterior Farey graphs with all added points on two sides, opposite P_1 and P_2.

LEMMA 1. $S_c = \Sigma_{i+j=c-1} S_i S_j$.

PROOF. If $R_1R_2R_3$ is a simplex in a graph F, then the number of graphs obtainable by c blowings up centered on the edge R_2R_3 also equals S_c, since we can apply the Farey map taking P_i to R_i for $i = 1, 2, 3$. The only possible center for the first blowing up is P_2P_3, that being, at the first stage, the only edge on the side opposite P_1. Let P_4 be the resulting center point.

If F is any graph obtained by blowing up n points on P_2P_3, then in addition to the point P_4 there must be i points on P_2P_4 and j points on P_4P_3, with $i + j = c - 1$. Since the number of possible blowings up of $P_1P_2P_4$ with i centers on P_2P_4 is S_i, and the number of possible blowings up of $P_1P_4P_3$ with j centers on P_4P_3 is S_j, we have S_iS_j possibilities for each pair i, j with $i + j = c - 1$, giving

$$S_n = \sum_{i+j=c-1} S_i S_j.$$

The first few values are as follows:

$$S_0 = 1,$$

$$S_1 = S_0S_0 = 1,$$

$$S_2 = S_0S_1 + S_1S_0 = 2,$$

$$S_3 = S_0S_2 + S_1S_1 + S_2S_0 = 5,$$

$$S_4 = S_0S_3 + \cdots + S_3S_0 = 14,$$

$$S_5 = S_0S_4 + \cdots + S_4S_0 = 42.$$

The combinatorist may recognize these as the Catalan numbers $(2c)!/c!(c+1)!$. However, we will derive a few more results from geometry before plunging into combinatorics.

LEMMA 2. *Let N_c be the number of Farey graphs with $c + 3$ vertices all lying on the sides opposite to P_1 and to P_2.*
Then $N_c = 2\Sigma_{i+j=c-1} N_i S_j$ for $c > 0$.

PROOF. The first blowing up must be either the edge P_2P_3 or the edge P_3P_1. Since the two cases are exactly symmetrical, and the first blowing up is uniquely determined by the line containing the center point $\frac{1}{3}(P_1+P_2+P_3)$, we will calculate the number starting with the blowing up P_2P_3, and then double it.

Again we consider the possibilities for the two simplices $P_1P_4P_3$ and $P_1P_2P_4$ separately (see Fig. 2). Assume that there are i points in the first and j points in

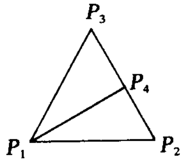


Fig. 2.

the second, with $i+j=c-1$. The number of possibilities for graphs obtained by blowing up edges opposite P_1 or P_4 in $P_1P_4P_3$ is N_i. Similarly the number of blowings up of $P_1P_2P_4$ with centers only in the edge opposite P_1 is S_j. Thus for each pair (i,j) we have a total of N_iS_j possibilities. Considering all pairs (i,j) and multiplying by 2 to cover symmetrical cases, we have, for $c>0$,

$$N_c = 2 \sum_{i+j=c-1} N_iS_j.$$

Again we compute the first few terms

$$\begin{aligned} N_0 &= 1, & S_0 &= 1,\\ N_1 &= 2N_0S_0 = 2, & S_1 &= 1,\\ N_2 &= 2(N_0S_1+N_1S_0) = 6, & S_2 &= 2,\\ N_3 &= 2(N_0S_2+\cdots+N_2S_0) = 20, & S_3 &= 5,\\ N_4 &= 2(N_0S_3+\cdots+N_3S_0) = 70, & S_4 &= 14. \end{aligned}$$

The reader may notice the following pattern:

COROLLARY. $N_m = (m+1)S_m$.

PROOF. By induction. For $m=0$, we have $N_0=1$. So assume the formula to be true for $m<k$.

$$N_k = 2(N_0S_{k-1}+\cdots+N_{k-1}S_0).$$

Case 1. $k = 2c + 1$.

$$\begin{aligned}N_k &= 2[(N_0S_{k-1} + N_{k-1}S_0) + (N_1S_{k-2} + N_{k-2}S_1) + \cdots + (N_cS_c)]\\ &= 2[(S_0S_{k-1} + kS_{k-1}S_0) + (2S_1S_{k-2} + (k-1)S_{k-2}S_1) + \cdots + (c+1)S_cS_c]\\ &= 2\left[(1+k)S_0S_{k-1} + (1+k)S_1S_{k-2} + \cdots + (1+k)S_{c-1}S_{c+1} + \frac{k+1}{2}S_cS_c\right]\\ &= (1+k)[2S_0S_{k-1} + \cdots + 2S_{c-1}S_{c+1} + S_cS_c]\\ &= (1+k)\sum_{i+j=k-1} S_iS_j\\ &= (1+k)S_k.\end{aligned}$$

Case 2. $k = 2c$.

$$\begin{aligned}N_k &= 2[(N_0S_{k-1} + N_{k-1}S_0) + \cdots + (N_{c-1}S_c + N_cS_{c-1})]\\ &= 2[(1+k)S_0S_{k-1} + \cdots + (1+k)S_{c-1}S_c]\\ &= (1+k)\sum_{i+j=k-1} S_iS_j\\ &= (1+k)S_k.\end{aligned}$$

LEMMA 3. $M_c = 3\sum_{i+j=c-1} N_iN_j$.

PROOF. As before, there is a unique first blowing up, determined by the line containing the midpoint $\frac{1}{3}(P_1 + P_2 + P_3)$. Since the 3 sets of graphs with different first blowings-up are disjoint, and symmetrical under rotation, the total number of possibilities is three times the number of graphs whose first center is the edge P_2P_3. By a now familiar process, we presume that of the remaining $c - 1$ points, i lie in the simplex $P_1P_4P_3$ and j in the simplex $P_1P_2P_4$. Since in each of these simplices the blowings up occur only on two of the three edges, those which are segments of the basic graph G_0, we have N_iN_j possibilities for each pair (i, j), giving

$$M_c = 3\sum_{i+j=c-1} N_iN_j.$$

Again we calculate the first few terms

$$\begin{aligned}M_1 &= 3,\\ M_2 &= 3(1\cdot 2 + 2\cdot 1) = 3\cdot 4,\\ M_3 &= 3(1\cdot 6 + 2\cdot 2 + 6\cdot 1) = 3\cdot 16,\\ M_4 &= 3(1\cdot 20 + 2\cdot 6 + 6\cdot 2 + 20\cdot 1) = 3\cdot 64.\end{aligned}$$

The conjecture $M_c = 3 \cdot 4^{c-1}$ presents itself immediately. In lieu of the geometrical proof we might have preferred, but have not found, we give a combinatorial proof:

Before obtaining formulae for M_c, we must first complete S_c and N_c. The standard combinatorial technique for determining a sequence of recursively defined numbers begins by constructing the formal power series with coefficients from the sequence. This series is called the generating function. Let us define two generating functions

$$f(x) = S_0 + S_1x + S_2x^2 + S_3x^3 + \cdots$$

and

$$g(x) = N_0 + N_1x + N_2x^2 + N_3x^3 + \cdots.$$

As an illustration of the utility of the generating function, we recall that by the corollary to Lemma 2, $N_c = (c+1)S_c$. Substituting in the formula for $g(x)$, we have

$$g(x) = S_0 + 2S_1x + 3S_2x^2 + \cdots.$$

One need not be a combinatoralist to notice that $g(x) = (xf(x))'$.

We now search for an analytic expression for $f(x)$, from which the coefficients can be recovered via Taylor expansions. We then try to use the recursion formula for the coefficients to establish a "functional equation", some expression in $f(x)$ and x which is identically zero.

In order to calculate S_c, the number of Farey graphs with c added points all on a given edge P_2P_3, we associate to each a tree, on which the vertices represent segments of P_2P_3. After the first blowing-up, P_2P_3 becomes a branch point, and we attach to it two terminal vertices representing P_2P_4 and P_4P_3. After i blowings up we will have $i+1$ terminal vertices corresponding from left to right to the segements of the Farey graph. In addition there will be i branch points, each with two branches coming out of it, corresponding to the segments at intermediate stages in the factorization. If the i-th step consists of blowing up the k-th segment and replacing it by two segments, then we transform the tree by converting the k-th terminal vertex to a branch point, attaching two new terminal vertices to it. As an illustration, all binary trees for $c = 4$ are given at the end of the paper in Example 4.

After noting this correspondence between graphs and trees, the authors applied the theory of DeBruijn [2] for the enumeration of "tree-shaped molecules" to calculate the functional equation of $f(x)$. In retrospect this was somewhat like shooting a fly with an anti-aircraft missile since the trees corresponding to the S_i are simple binary trees whose enumeration is one of the better known facts of combinatorics. For that reason it seems unnecessary to describe the method originally used, and we will rely on the fact that constructing functional equations is an art form similar to integration in elementary calculus: once one has the result it is necessary only to substitute $f(x)$ in order to prove that it is correct.

LEMMA 4. *The generating function*

$$f(x) = S_0 + S_1 x + S_2 x^2 + \cdots$$

satisfies the functional equation

$$xf^2(x) - f(x) + 1 = 0.$$

Thus

$$f(x) = \frac{1}{2x}(1 - \sqrt{1-4x}),$$

and therefore $S_c = (2c)!/c!(c+1)!$.

PROOF. We define a generating function

$$f(x) = S_0 + S_1 x + S_2 x^2 + \cdots.$$

Squaring, we have

$$\begin{aligned} f^2(x) &= S_0^2 + (S_0 S_1 + S_1 S_0)x + \cdots + \left(\sum_{i+j=c} S_i S_j\right) x^c + \cdots \\ &= S_1 + S_2 x + S_3 x^2 + \cdots \\ &= \frac{f(x)-1}{x}. \end{aligned}$$

This gives a functional equation

$$xf^2(x) - f(x) + 1 = 0,$$

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Of the two possible solutions, only the one with the negative sign gives the right initial terms, so

$$f(x) = \frac{1}{2x}(1 - \sqrt{1-4x}).$$

Using the extended binomial expansion,

$$(1-4x)^{1/2} = 1 + \frac{\frac{1}{2}}{1!}(-4x) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}(-4x)^2 + \cdots + \frac{\frac{1}{2}\cdot(-\frac{1}{2})\cdots\left(\frac{1-2c}{2}\right)}{(c+1)!}(-4x)^{c+1} + \cdots$$

$$= 1 - \left(\frac{1}{1!}(2x) + \frac{1\cdot 1}{2!}(2x)^2 + \cdots + \frac{1\cdot 1\cdot 3\cdots(2c-1)}{(c+1)!}(2x)^{c+1} + \cdots\right);$$

$$f(x) = \frac{1}{2x}(1-(1-4x)^{1/2})$$

$$= \sum_{c=0}^{\infty} \frac{1\cdot 3\cdots 2c-1}{(c+1!)}(2x)^c$$

$$= \sum_{c=0}^{\infty} \frac{(2c)!}{(c+1)!\,c!}x^c.$$

LEMMA 5. *If $g(x) = N_0 + N_1 x + \cdots$ is the generating function for $\{N_i\}$, then*

$$g(x) = (1-4x)^{1/2},$$

$$N_c = (c+1)S_c = \frac{(2c)!}{c!\,c!}.$$

PROOF.

$$g(x) = N_0 + N_1 x + N_2 x + \cdots$$

$$= S_0 + 2S_1 x + 3S_2 x^2 + \cdots$$

$$= (xf(x))'$$

$$= \tfrac{1}{2}(1-(1-4x)^{1/2})'$$

$$= \tfrac{1}{4}(-1)(1-4x)^{-1/2}(-4)$$

$$= (1-4x)^{-1/2}.$$

We could calculate N_c from the Taylor expansion of $g(x)$, but it is simpler to combine Lemma 2 and Lemma 4:

$$N_c = (c+1)S_c = (c+1)\frac{(2c)!}{(c+1)!\,c!} = \frac{(2c)!}{c!\,c!}.$$

LEMMA 6. *If $h(x) = M_1 + M_2 x + M_3 x^2 + \cdots$, then $h(x) = 3(1-4x)^{-1}$. Thus* $M_c = 3\cdot 4^{c-1}$.

PROOF. Noting the similarity between the formula for calculating S_c and the formula for calculating M_c, we see that

$$\begin{aligned} g^2(x) &= N_0N_0 + (N_0N_1 + N_1N_0)x + \cdots \\ &= \tfrac{1}{3}(M_1 + M_2x + M_3x^2 + \cdots) \\ &= \tfrac{1}{3}h(x). \end{aligned}$$

Thus

$$\begin{aligned} h(x) &= 3g^2(x) \\ &= 3 \cdot ((1-4x)^{-1/2})^2 \\ &= 3(1-4x)^{-1} \\ &= 3(1 + 4x + (4x)^2 + \cdots). \end{aligned}$$

To complete the proof of the proposition it remains to associate a weighted binary tree to each exterior graph, and to show that the number of alternative factorizations depends only on this tree. The root branch point will represent the first blowing up, and hanging from each of the two branches will be the trees of the corresponding subgraphs. Each branch point will be weighted by a number between one and three representing the number of edges of the corresponding subtriangle which are available for blowing up. If both new simplices have the same number of exterior edges, the one in the clockwise direction will be on the left. If not, then the one with the larger number of exterior edges will go on the left.

We claim that the number of exterior Farey graphs associated to a given binary tree is the product of the weights of the branch points. At the root there are three possible sides to be blown up. This choice produces three disjoint classes, which are isomorphic under rotation. The blowing-up bisects one of the corners. Thereafter the remaining two corners remain empty, because the sides opposite to them are not exterior. Thus after c blowings up there will be two corner simplices and $c-1$ wedge simplices containing only one exterior edge. The corner simplices can be blown up in one of two ways, and thus are given a weight of two. The blowing-up produces one corner simplex and one new wedge. The wedge simplices have a unique blowing up and produce only wedges.

The number of blowings-up centered in corner simplices is the sum of the lengths of the leftmost path from each of the two original branches. If this number is m, then the total number of possibilities in each of the three classes is

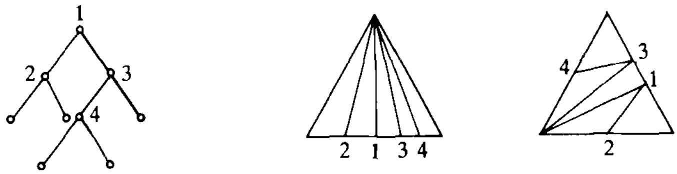


Fig. 3.

2^m, giving $3 \cdot 2^m$ possibilities altogether. See Fig. 3 for two examples of graphs sharing a binary tree of weight $3 \cdot 2^3 = 24$.

To complete the proof of the proposition, we need only show that the number of alternative factorizations depends only on the binary tree. We will in fact prove the following explicit formula:

LEMMA 7. *Suppose that to each branch vertex v_i, $i = 1, \cdots, c$ in the binary tree associated with an exterior Farey graph F, we associate numbers l_i and r_i, giving the number of branch points in the subtrees hanging from the left and right branches, respectively. Then the number of alternative factorizations of F is*

$$\prod_{i=1}^{c} \frac{(l_i + r_i)!}{l_i!\, r_i!}\ .$$

PROOF. Let us represent the number of alternative factorizations of a graph F by $P(F)$. If G is a simplex, with no branch points in the corresponding tree, then $P(F) = 1$. We proceed by induction, assuming the lemma is true for $c - 1$. (The product is empty and equal to 1 if $c - 1 = 0$.)

The first blowing-up in F is uniquely determined, giving two subgraphs F_l and F_r. The tree of F has l_1 branch points and the tree of F_r has r_1 branch points. If $a_1, \cdots, a_{l_1}$ is one of the $P(F_l)$ factorization sequences for F_l, and $b_1, \cdots, b_{r_1}$ is one of the $P(F_r)$ factorization sequences for F_r, then the number of ways of interspersing these two sequences into one long sequence $c_1, \cdots, c_{l_1+r_1}$ is

$$\binom{l_1 + r_1}{r_1} = \frac{(l_1 + r_1)!}{l_1!\, r_1!}\ ,$$

the number of ways of choosing r_1 of the c_j to hold the r_1 elements of the b_i sequence. Thus altogether we have

$$P(F) = \frac{(l_1 + r_1)!}{l_1!\, r_1!}\, P(F_l) P(F_r);$$

substituting for $P(F_l)$ and $P(F_r)$ by induction, we get the desired result.

We calculate $P(F)$ for three different graphs with seven branch points. Since the number depends only on the tree, we may as well write $P(T)$. We also calculate $M(T)$, $N(T)$ and $S(T)$, the number of three-sided, two-sided and one-sided graphs sharing that tree. The calculation of $M(T)$ was given earlier. $N(T)$ is obtained by weighting only the branch points on the leftmost path by 2, and for $S(T)$ every branch point has weight 1, so that there is a one-to-one correspondence between binary trees and one sided graphs. See Fig. 4.

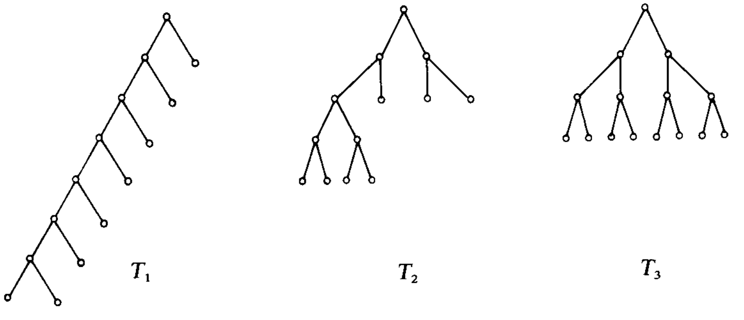


Fig. 4.

EXAMPLE 1. $T = T_1$

$$P(T)=\binom{6}{6}\binom{5}{5}\binom{4}{4}\binom{3}{3}\binom{2}{2}\binom{1}{1}\binom{0}{0}=1,$$

$$M(T)=3\cdot 2^6=192,$$

$$N(T)=2^7=128,$$

$$S(T)=1.$$

EXAMPLE 2. $T = T_2$

$$P(T)=\binom{6}{5}\binom{4}{4}\binom{3}{1}\binom{0}{0}\binom{1}{1}\binom{0}{0}\binom{0}{0}$$

$$=6\cdot 3$$

$$=18,$$

$$M(T)=3\cdot 2^3\cdot 2=48,$$

$$N(T)=2^4=16,$$

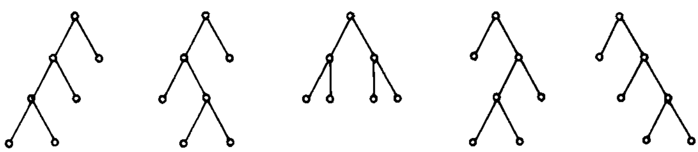
$$S(T)=1.$$

EXAMPLE 3. $T = T_3$

$$P(T)=\binom{6}{3}\binom{2}{1}\binom{0}{0}\binom{0}{0}\binom{2}{1}\binom{0}{0}\binom{0}{0}$$
$$=\frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}\cdot 2\cdot 2$$
$$=80,$$
$$M(T)=3\cdot 2^2\cdot 2^2=48,$$
$$N(T)=2^3=8,$$
$$S(T)=1.$$

Note that $M_7 = 3\cdot 4^6$, while the total number of factorization sequences with seven blowings up is $9!/2$. The average is thus around 29.5. Since the binomial coefficients appearing in the formula for $P(F)$ are largest when $l_i = r_i$, the symmetrical graph in Example 3 gives the maximum value for $P(T)$ for $c = 7$.

EXAMPLE 4. We give the complete catalogue of trees for $c = 3$ (Fig. 5).



T_1 T_2 T_3 T_4 T_5

Fig. 5.

$P(T_i)$	1	1	2	1	1
$M(T_i)$	12	6	12	12	6
$N(T_i)$	8	4	4	2	2
$S(T_i)$	1	1	1	1	1

$$\sum M(T_i)=M_3=48,\qquad \sum P(T_i)M(T_i)=\frac{(3+2)!}{2!}=60,$$

$$\sum N(T_i)=N_3=20,\qquad \sum P(T_i)N(T_i)=\frac{(3+1)!}{1!}=24,$$

$$\sum S(T_i)=S_3=5,\qquad \sum P(T_i)S(T_i)=3!=6.$$

EXAMPLE 5. Without actually drawing the trees, we give in tabular form the corresponding numbers for the fourteen trees with $c = 4$.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$P(T_i)$	1	1	2	1	1	3	3	3	3	1	1	2	1	1
$M(T_i)$	24	12	12	6	6	24	12	24	12	24	12	12	6	6
$N(T_i)$	16	8	8	4	4	8	4	4	4	2	2	2	2	2
$S(T_i)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1

$$\sum M(T_i) = 192, \qquad \sum P(T_i)M(T_i) = \frac{6!}{2!} = 360,$$

$$\sum N(T_i) = 70, \qquad \sum P(T_i)N(T_i) = \frac{5!}{1!} = 120,$$

$$\sum S(T_i) = 14, \qquad \sum P(T_i)S(T_i) = 4! = 24.$$

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DEPARTMENT OF MATHEMATICS
BAR-ILAN UNIVERSITY
RAMAT GAN 52100, ISRAEL